Vortices, circumfluence, symmetry groups and Darboux transformations of the Euler equations

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The Euler equation (EE) is one of the basic equations in many physical fields such as the fluids, plasmas, condense matters, astrophysics, oceanic and atmospheric dynamics. A new symmetry group theorem of the two dimensional EE is obtained via a simple direct method and the theorem is used to find *exact analytical* vortex and circumfluence solutions. Some types of Darboux transformations (DTs) for the both two and three dimensional EEs are obtained for *arbitrary spectral parameters* which indicates that the EEs are integrable and the Navier-Stockes (NS) equations with large Renoyed number are nearly integrable, i.e, they are singular perturbations of the integrable EEs. The possibility of the vortex and circumfluence solutions to approximately explain the tropical cyclones (TCs), especially, the Hurricane Andrew 1992, is discussed.

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1. Introduction. In fluid physics, there are various important open problems. One of the most important of them is the existence and smoothness problem of the NS equation which is the basic equation and the starting point of all the problems in fluid physics [1]. Due to its importance and difficulty, the problem is listed as one of The Millennium Problems of the 21st century [2].

One of the most significant recent development related to this problem may be the discovery of the weak Lax pairs of the two and three dimensional EEs which are the limit cases of the NS equation for the large Renoyed number [3]. That means the EEs are integrable and thus the NS equation with large Renoyed number is nearly integrable, the singular perturbation of integrable models!

The (3+1)-dimensional EE

$$E_1 \equiv \vec{\omega}_t + (\vec{u}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{u} = 0,\ \vec{\omega} = \nabla\times\vec{u}, \quad (1)$$

with $\nabla\cdot\vec{u} = 0$ is the original springboard for investigating the incompressible inviscid fluid. In the equation system (1), $\vec{\omega} \equiv \{\omega_1,\ \omega_2,\ \omega_3\}$ is the vorticity and $\vec{u} \equiv \{u_1,\ u_2,\ u_3\}$ is the velocity of the fluid. In (2+1)-dimensional case, the EE is simplified to ($[\psi,\omega] \equiv \psi_x\omega_y - \psi_y\omega_x$)

$$E \equiv \omega_t + [\psi,\omega] = 0,\ \omega = \psi_{xx} + \psi_{yy}, \quad (2)$$

where the velocity $\vec{u} = \{u_1,\ u_2\}$ is linked to the stream function ψ by $u_1 = -\psi_y,\ u_2 = \psi_x$.

The EEs are important not only in fluid physics [4] but also in many other physical fields such as the plasma physics [5], oceanography [6], atmospheric dynamics [7], superfluid and superconductivity [8], cosmography and astrophysics [9], statistical physics [10], field and particle physics[11] and condense matters such as the Bose-Einstein condensation [12], the crystal liquid [13] and the liquid metallic hydrogen [14] etc.

There are innumerable papers on the EEs in literature due to EE (1) or (2) is just the beginning point to study various physical problems. However, most of them only treat the equation approximately or numerically because the weakly Lax integrability of the model has just been revealed most recently by Charles Li [3]. So there is few results on the *exact* and *analytic* solutions of the EEs.

It is known that to find some exact and analytic results of a physical system, Lie group theory is one of the most effective methods. Nevertheless, even for a mathematician, to find the symmetry group of a given nonlinear system is still very difficult especially for non-Lie and non-local symmetry groups. So it is significant for a physicist to find a *simple* method to get *more general* symmetry groups of nonlinear systems without use of the complicated group theory.

Vortices and circumfluence are most general observations in some physical fields especially for fluid systems. In our knowledge, there is little exact analytic understanding on vortices and circumfluence though there are rich vortex structures for different physical systems. If one can find the full symmetry group of the EE, one may get some kinds of exact vortex and circumfluence solutions from some simple trivial or periodic seed solutions.

Though the (weak) Lax pairs of the EEs were found in both two and three dimensions four years ago, little further understanding of the exact solutions has been obtained from their Lax pairs. For the two-dimensional EE, one special type of DT with zero spectral parameter had also been given by Li [3]. In order to get nontrivial and as many as possible results from DT, one has to find DT with *nonzero* spectral parameter which is still failed for both two and three dimensional situations.

In this Letter, we firstly establish a simple direct method to find a general group transformation theorem for the two dimensional EE. Then the theorem is used to find a quite general symmetric vortex and/or circumfluence solution with some arbitrary functions starting from a quite trivial periodic solution (Rosbby wave). Some velocity field plots related to vortices and circumfluence are

given. The exact vortices and the circumfluence solutions are used to explain TC eye, track and the relation between track and the background wind. Some DTs with *arbitrary spectral parameters* for two and three dimensional EEs are explicitly given.

2. Space-time transformation group of the two dimensional EE. In the traditional theory, to find the Lie symmetry group of a given nonlinear physical system, one has to firstly find its Lie symmetry algebra and then use the First Fundamental Theorem to solve an "initial" problem. If one utilize the standard Lie group theory to study the symmetry group of the two-dimensional EE, it is easy to find that the only possible symmetry transformations are: the arbitrary time dependent space and stream translations, constant time translation, space rotation and scaling transformations. The details on the traditional Lie point symmetries can be found in the literature, say, [15].

Recently, it is found that for simplicity and to find *more general* symmetry groups, one may use some types of new simple direct method especially for Lax integrable models without any group theory [16, 17].

For the two-dimensional EE, its weak Lax pair reads

$$\omega_x\phi_y - \omega_y\phi_x = \lambda\phi,\ \phi_t + \psi_x\phi_y - \psi_y\phi_x = 0, \quad (3)$$

where λ is the spectral parameter. That means the compatibility condition of (3) is just the EE (2). We say the Lax pair is weak because that (1) is a compatibility condition only under the meaning that $[E,\ \phi] = 0$ [3].

It is not difficult to understand that the symmetry group of the EE can be obtained by using the following gauge transformation companied with the space time transformation for the spectral function ϕ

$$\phi \to g\phi'(\xi,\ \eta,\ \tau) \equiv g\phi', \quad (4)$$

where $g,\ \xi,\ \eta$ and τ are undetermined functions of $\{x,\ y,\ t\}$. The function ϕ' in (4) should satisfy the same Lax pair (3) but with different independent variables, i.e.,

$$\omega'_\xi\phi'_\eta - \omega'_\eta\phi'_\xi = \lambda'\phi',\ \phi'_\tau + \psi'_\xi\phi'_\eta - \psi'_\eta\phi'_\xi = 0, \quad (5)$$

where $\{\omega',\ \psi'\}$ is also a solution of the EE (2) under the transformation $\{x,\ y,\ t,\ \lambda\} \to \{\xi,\ \eta,\ \tau,\ \lambda'\}$. Substituting the gauge transformation (4) into the Lax pair (3), eliminating ϕ'_η and ϕ'_τ by means of (5) and vanishing the coefficients of ϕ'_ξ and ϕ' of the resulting equations, we have the determining equations for the functions $\xi,\ \eta,\ \tau,\ \psi',\ \omega'$ and g:

$$\begin{aligned}
&\{([\tau,\omega]\psi'_\xi + [\omega,\eta])\lambda' - \lambda\omega'_\xi\}g + [\omega,g]\omega'_\xi = 0,\\
&\{\eta_t + [\psi,\eta] - (\tau_t + [\psi,\tau])\psi'_\xi\}\lambda' g + (g_t + [\psi,g])\omega'_\xi = 0,\\
&(\tau_t + [\psi,\tau])\omega'_\tau + (\eta_t + [\psi,\eta])\omega'_\eta + (\xi_t + [\psi,\xi])\omega'_\xi = 0,\\
&[\tau,\omega]\omega'_\tau + [\omega,\eta]\omega'_\eta + [\omega,\xi]\omega'_\xi = 0. \quad (6)
\end{aligned}$$

It is clear that there are abundant interesting exact solutions because the determining equation system (6) is under-determined. Here we just write down one special solution theorem of the two dimensional EE.

Theorem 1. *If $\{\omega(x,\ y,\ t),\ \psi(x,\ y,\ t)\}$ is a known solution of the two dimensional EE, so is $\{\omega' \equiv F(\omega(\xi,\ \eta,\ \tau),\ t) \equiv F,\ \psi' \equiv G(\omega(\xi,\ \eta,\ \tau),\ t) + f \equiv G + f\}$ with $f = f(x,\ y,\ t),\ \xi = \xi(x,\ y,\ t),\ \eta = \eta(x,\ y,\ t)$ and $\tau \equiv \tau(t)$ are functions of the indicated variables while the six functions $\{f,\ \xi,\ \eta,\ \tau,\ F,\ G\}$ need to satisfy two constrained equations*

$$\begin{aligned}
&F = G_{\omega\omega}(\omega_x^2 + \omega_y^2) + G_\omega(\omega_{xx} + \omega_{yy}) + f_{xx} + f_{yy},\\
&f_x\omega_y - \omega_x f_y + \omega_t = 0, \quad (7)
\end{aligned}$$

where $G_\omega = \partial G/\partial\omega$ and $\omega_x,\ \omega_y$ and ω_t are total derivatives, say, $\omega_t = \omega_\xi\xi_t + \omega_\eta\eta_t + \omega_\tau\tau_t$.

Starting from some simple seed solutions, one can obtain many physically interesting solutions from Theorem 1. For instance, if we take the seed solution as the usual Rossby wave solution

$$\begin{aligned}
&\psi = ax - by + c\cos(kx + ly + (ka + bl)t),\\
&\omega = -(k^2 + l^2)c\cos(kx + ly + (ka + bl)t), \quad (8)
\end{aligned}$$

then we can get the following new general vortex and/or circumfluence solution ($r \equiv (x - x_0)^2 + (y - y_0)^2$):

$$\begin{aligned}
\psi' &= y_{0t}x - x_{0t}y + F_1\ln r + F_2\\
&\quad -\frac{1}{2}h_{0t}\tan^{-1}\frac{x - x_0}{y - y_0} + \frac{1}{4}\int\frac{F(r + h_0)}{r}\mathrm{d}r,\\
\omega' &= F_r(r + h_0), \quad (9)
\end{aligned}$$

where $x_0,\ y_0,\ h_0,\ F_1$ and F_2 are arbitrary functions of t and $F \equiv F(r + h_0)$ is an arbitrary function of $r + h_0$.

Because of the intrusion of the many arbitrary functions into the exact solution (9), one is able to find various vortex and circumfluence structures by selecting the arbitrary functions in different ways. Here are some special examples for $h_0 = F_1 = F_2 = 0$.

3. Lump-type vortices. If we take the function $F(r)$ as a rational solution of r , $F(r) = \frac{\sum_{i=0}^N a_i r^i}{\sum_{i=0}^N b_i r^i} \equiv \frac{P(r)}{Q(r)}$, with the conditions $b_N \neq 0$ and $Q(r) \neq 0$ for all $r \geq 0$, then the solution (9) becomes an analytical lump-type vortex and/or circumfluence solution for the velocity field. For instance, Fig. 1a shows a typical lump-type vortex solution for the velocity field with the selection, $F(r) = -4r^2(1 + r)^{-2}$, at times for $x_0 = y_0 = 0$. The related velocity possesses the rational form

$$u = 2(y - y_0)r(1 + r)^{-2},\ v = 2(x_0 - x)r(1 + r)^{-2}. \quad (10)$$

4. Dromion-type vortices. If we take the function $F(r)$ as a rational function of r multiplied by an exponentially decayed factor, say, $\exp(-r)$, then (9) becomes an analytical dromion-type vortex and/or a circumfluence solution. Fig. 1b shows a typical dromion-type vortex solution for the velocity field with

$$F(r) = 100re^{-r}, \quad (11)$$

at times t_i for $x_0(t_i) = y_0(t_i) = 0$.

5. Ring soliton solutions and circumfluence. In the recent studies of (2+1)-dimensional nonlinear physics systems, some kinds of ring soliton solutions have been found

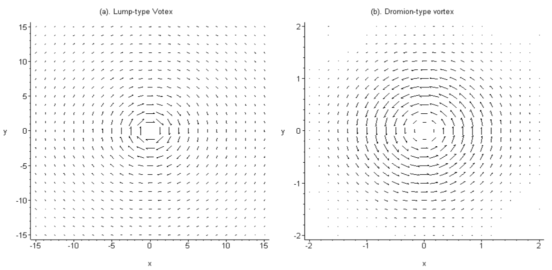

FIG. 1: (a). A field plot of the typical lump-type vortex given by (10) with $x_0 = y_0 = 0$. (b). A field plot of the dromion-type vortex with the selection (11) for $x_0 = y_0 = 0$.

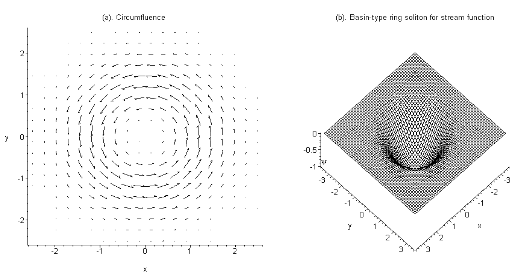

FIG. 2: (a). A field plot of the circumfluence with the selection (12) and $x_0 = y_0 = 0$ for the velocity $\{u,\ v\}$. (b). A special basin type ring soliton for the stream function ψ related to (a).

[18, 19]. It is interesting that the basin-type and/or plateau-type of soliton solutions may be responsible for the circumfluence solution for the fluid system described by EE. For instance, if we select the function $F(r)$ possesses the property $d^iF(r)/dr^i\big|_{r=0} = 0,\ i = 0,\ 1,\ ...,\ n$ for $n \geq 2$, then (9) expresses the circumfluence for the velocity field and the basin-type or plateau type ring soliton for the stream function. Fig. 2 is a special plot of (9) with the selections $x_0 = y_0 = 0$ and

$$F(r) = -4r^2e^{-r}. \qquad (12)$$

Fig. 2a exhibits the circumfluence structure for the velocity field related to the selection (12) and Fig. 2b shows the basin type ring soliton shape for the stream function ψ with the same selections as Fig. 2a.

6. Tropical TC track and background wind. From the expression of the exact solution (9) and the Figs. 1–2, we know that the vortices in physics may have quite rich structures. Due to the richness of the solution structures and wide applications of the vortex in various fields such as the fluids, plasma, oceanic and atmospheric dynamics, cosmography, astrophysics, condense matters etc [4]–[14], the results may be applied in all of these fields. For instance, in the oceanic and atmospheric dynamics, the analytical solutions expressed by (9) may be used to approximately describe the TCs which possess increasing destructiveness over the past 30 years [20]. The relatively tranquil part, the center of the circumfluence shown by Fig. 2 is responsible for the TC eye [21]. Furthermore, the expression (9) also offers a relation between the TC track given by $\{x_0(t),\ y_0(t)\}$ and the strength of the background wind described by $\{x_{0t},\ y_{0t}\}$. Fig. 3 shows an example on the TC track (Fig. 3b) and the related background wind field (Figs. 3c–3f). To compare with real observations, the track of the Hurricane Andrew 1992 from 23 to 27 August is also plotted at Fig. 3a. From Fig.3 one can find that the background wind lead to the change of the direction of the TC track.

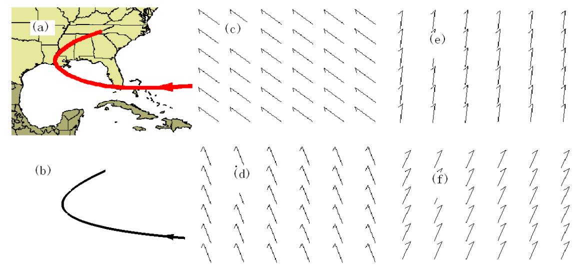

FIG. 3: (a) The real TC track of the Hurricane Andrew 1992 from 23 to 27 August. (b) The theoretical track given by $x_0 = 0.1(t-3.7)^2 - 3,\ y_0 = 0.1t^2 - 3$ from day 1 to day 5. (c)–(f) The background wind fields related to the TC track (b) at day 2, 3, 4 and 5 respectively.

7. DTs of EEs. In [3], Li found a DT of EE (2) with Lax pair (3) for *zero* spectral parameter. For the three-dimensional EE, though two types of weak Lax pairs had also been given by Li et al, there is no DT found even for zero spectral parameter. Here we write down one new DT for the two-dimensional EE (2) and two DT theorems for the three-dimensional EE (1) with *arbitrary spectral parameters* without any proof for lacking of space.

Theorem 2. *If $\{\omega,\ \psi,\ \phi\}$ is a solution of* (2) *and* (3) *with the spectral parameter λ, $G(f)$ being an arbitrary function of f that is a given spectral function of* (3) *under the spectral parameter λ_0, then $\{\omega + q,\ \psi + p,\ G\phi\}$ with the spectral parameter λ_1 is also a solution where p and q are determined by $q \equiv p_{xx} + p_{yy}$ and*

$$[p,\ G\phi] = 0,\ [q,\ G\phi] = (\lambda_1 - \lambda)G\phi - \lambda_0 G_f. \qquad (13)$$

It is interesting that if we take all the parameters λ, λ_0 and λ_1 as zero, the Bäcklund transformation $\omega' = \omega + q$, $\psi' = \psi + p$ with (13) is equivalent to that obtained by Li [3]. To see this point more clearly, one can write (13) in an alternative form by eliminating ϕ_y via Lax pair (3)

$$\begin{aligned}&\lambda_1\omega_x - [\lambda + \lambda_0 f(\ln G)_f](\omega + q)_x + [\omega,\ q][\ln(G\phi)]_x = 0,\\ &[\omega + q,\ p][\ln(G\phi)]_x + \lambda_1 p_x = 0.\end{aligned} \qquad (14)$$

Theorem 3. *If $\{\vec{\omega},\ \vec{u},\ \phi\}$ is a solution of EE* (1) *and its scalar Lax pair [3] $\{\vec{\omega}\cdot\nabla\phi = \lambda\phi,\ \phi_t + \vec{u}\cdot\nabla\phi = 0\}$, so is $\{\vec{\omega}+\vec{q},\ \vec{u}+\vec{v},\ G\phi\}$ with the spectral parameter λ_1, where G is an arbitrary spectral function of f under the spectral parameter λ_0 while $\vec{v}$ and $\vec{q}$ are determined by $\phi G_f\vec{q}\cdot\nabla f + G\vec{q}\cdot\nabla\phi + \lambda_0 f\phi G_f + (\lambda - \lambda_1)G\phi = 0$ and $\phi G_f\vec{v}\cdot\nabla f + G\vec{v}\cdot\nabla\phi = 0$.*

Theorem 4. *If $\{\vec{\omega},\ \vec{u},\ \vec{\phi}\}$ is a solution of EE* (1) *and its vector Lax pair [3]$\{\vec{\omega}\cdot\nabla\vec{\phi} - \vec{\phi}\cdot\nabla\vec{\omega} = \lambda\vec{\phi},\ \vec{\phi}_t + \vec{u}\cdot\nabla\vec{\phi} -$*

$\vec{\phi}\cdot\nabla\vec{u}=0,\}$, *so is* $\{\vec{\omega}+\vec{q},\ \vec{u}+\vec{v},\ \vec{f}\diamond\vec{\phi}\}$ *with the spectral parameter* λ_1 *and* $\vec{f}$ *being a given spectral vector function under the spectral parameter* λ_0 *while* $\vec{v}$ *and* $\vec{q}$ *determined by* $(\vec{q}\cdot\nabla)(\vec{f}\diamond\vec{\phi})-[(\vec{f}\diamond\vec{\phi})\cdot\nabla]\vec{q}=(\lambda_1-\lambda-\lambda_0)\vec{f}\diamond\vec{\phi}$ *and* $(\vec{v}\cdot\nabla)(\vec{f}\diamond\vec{\phi})-[(\vec{f}\diamond\vec{\phi})\cdot\nabla]\vec{v}=0$, *where "*$\diamond$*" can be any kinds of possible vector product (any possible map from* $R^3\times R^3\to R^3$*) as long as the Lebnitz differential rule is valid, i.e.,* $(\vec{a}\diamond\vec{b})_x=\vec{a}_x\diamond\vec{b}+\vec{a}\diamond\vec{b}_x$.

The simplest example of "$\diamond$" may be "$\times$", the usual cross product of vectors.

In principle, the repeated uses of the DT theorems 2–4 and some known seed solutions, one may get infinitely many new solutions of the two and three dimensional EEs. In real applications, it is still very difficult. For instance, though a special DT with zero spectral parameter was given four years ago, no concrete new solutions have been obtained from it up to now. Here we just offer a special concrete example for the two-dimensional EE.

It is obvious that the constant flow $u_1=b,\ u_2=a$, i.e., $\psi=ax-by,\ \omega=0$, is a solution of the two-dimensional EE. This trivial solution corresponds to $\lambda=\lambda_0=0$. Starting from this simple seed, Theorem 2 leads to a quite general but complicated solution with some arbitrary functions. It is interesting that the usual Rossby wave (8) is just a special case if we take $\lambda_1=0$. That means the Rossby wave (8) can be obtained as a special case of the first step DT with zero spectral parameters.

8. Summary and discussion. The analytical and exact forms of the vortices and circumfluence of the two dimensional fluid are studied by means of the symmetry group theorem 1 and the DT theorem 2 of the two dimensional EE. Starting from the trivial constant flow, the periodic Rossby wave are obtained as a special case of the first step DT. Using the group transformation theorem to the Rossby wave, a solution with some arbitrary functions can be obtained. The solution including many kinds of possible vortices and circumfluence such as the lump type vortices, dromion type vortices etc. The vortex and circumfluence solutions may be applied in various physics fields mentioned in the introduction section and the references [4]–[14]. Especially, they can be applied to approximately explain some important problems of TCs such as its eye, track and the relations between track and the background wind.

The DTs with arbitrary spectral parameters are successfully obtained. The existence of the DTs with *arbitrary spectral parameter* is important not only because infinitely many exact solutions of the EEs can be obtained but also the integrability of the EEs is confirmed. Hereafter the NS equations with large Renoyed number which are related to the usual important cases can be studied by means of the singular perturbations of the integrable ones. Consequently, the more general physical problems can be studied better. For instance, for the TC problems we can use the exact solutions of the EEs to study their steady areas near the eyes and the singular perturbation to investigate their chaotic parts, the areas around the eyes.

Because of the importance of the EEs and the NS system and their wide applications, the more about the models, their DTs and the exact solutions given in this Letter are worthy of further study.

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